Improved FPT Algorithms for Deletion to Forest-Like Structures

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Overview

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- FEEDBACK VERTEX SET(FVS): Given an undirected graph G, non-negative integer k, does there exist a subset $F \subseteq V(G)$ of size at most k s.t. $G - F$ is a forest?

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- Want FPT w.r.t k: $f(k) * \text{poly}(n)$
- Goal: Minimize function $f(k)$, $\text{poly}(n)$ factor does not matter.

Feedback Vertex Set - Prior Work

- Downey and Fellows '92: $f(k) = k^{\mathcal{O}(k)}$
- Iwata and Kobayashi [4, IPEC '19]: $f(k) = 3.46^k$

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- Li and Nederlof [6, SODA '20]: $f(k) = 2.7^k$

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- Generalizations of forests:
  - ℓ-forest: At most ℓ edges away from being a forest
  - *Pseudoforest*: Every connected component has at most one cycle

Independent FVS

INDEPENDENT FVS (IFVS)

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Given a graph G and a non-negative integer k, does there exist a fvs S of size at most k, that is also an *independent set* in G?

- Best known result: $\mathcal{O}^\star(3.619^k)$ (Li and Pilipczuk [7])

Almost Forest Deletion

ALMOST FOREST DELETION (AFD)

Given a graph G and two non-negative integers k and ℓ, does there exist a vertex subset S of size at most k such that $G - S$ is an ℓ-forest?

- Solution set: afd-set

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Generalized Almost Forest Deletion

Restricted Independent Almost Forest Deletion (RIAFD)

Given a graph G, a vertex set $R \subseteq V(G)$, and integers k and ℓ, does there exist a set $S \subseteq V(G)$ of size at most k that does not contain any element from R, that is also an independent set in G, and $G - S$ is an ℓ-forest?

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- Solution Set: riafd-set
- Vertices in $R \to$ Red vertices
- $R = \varnothing$ and $\ell = 0 \to$ IFVS
- Given instance of AFD:
  - Subdivide every edge
  - add all subdivision vertices to R

Pseudoforest Deletion

Pseudoforest Deletion (PDS)

Given a graph G and a non-negative integer k, does there exist a vertex subset S of size at most k such that $G - S$ is a pseudoforest?

Pseudoforest Deletion

PSEUDOFOREST DELETION (PDS)

Given a graph G and a non-negative integer k, does there exist a vertex subset S of size at most k such that $G - S$ is a pseudoforest?

- Best known result: $\mathcal{O}^\star(3^k)$(Bodlaender et al. [2])

Parameterized by Treewidth

Theorem (1)

There exists an $\mathcal{O}^\star(3^{\mathrm{tw}})$ *time Monte-Carlo algorithm that given a tree decomposition of the input graph of width* tw *solves the following problems:*

1. RESTRICTED-INDEPENDENT ALMOST FOREST DELETION *in exponential space.*
2. PSEUDOFOREST DELETION *in exponential space.*

Almost Forest Deletion (AFD)

Theorem (2)

There exist Monte-Carlo algorithms that solve RIAFD *problem in*

1. $\mathcal{O}^\star(3^k \cdot 3^\ell)$ *time and polynomial space.*
2. $\mathcal{O}^\star(2.85^k \cdot 8.54^\ell)$ *time and polynomial space.*
3. $\mathcal{O}^\star(2.7^k \cdot 36.61^\ell)$ *time and exponential space.*
4. $\mathcal{O}^\star(3^k \cdot 1.78^\ell)$ *time and exponential space.*

Independent FVS (IFVS)

Theorem (3)

There exist Monte-Carlo algorithms that solve INDEPENDENT FVS *in:*

1. $\mathcal{O}^\star(3^{\mathrm{tw}})$ *time, given a tree decomposition of width* tw.
2. $\mathcal{O}^\star(2.85^k)$ *time and polynomial space*
3. $\mathcal{O}^\star(2.7^k)$ *time and exponential space*

Pseudoforest Deletion Set (PDS)

Theorem (4)

There exists a Monte-Carlo algorithm that solves PSEUDOFOREST DELETION *in* $\mathcal{O}^\star(2.85^k)$ *time and polynomial space.*

Our Results

Ref	Problem	D/R	Complexity
[7]	IFVS	D	$\mathcal{O}^\star(3.619^k)$
[8]	AFD	D	$\mathcal{O}^\star(5^k 4^\ell)$
[2]	PDS	D	$\mathcal{O}^\star(3^k)$
Ours	IFVS	R	$\mathcal{O}^\star(2.7^k)$
	AFD	R	$\mathcal{O}^\star(\min\{2.85^k \cdot 8.54^\ell, 2.7^k \cdot 36.61^\ell, 3^k \cdot 1.78^\ell\})$
	PDS	R	$\mathcal{O}^\star(2.85^k)$

Framework

Framework of Li et al. [6].

Aim: Solution of size k in running time $\mathcal{O}^\star(\alpha^k)$

1. **Dense Case** (High avg. degree): Number of edges incident is $\Omega(k)$. Select a vertex in solution w.p. $\geq \frac{1}{\alpha}$.
2. Once dense case is done: k_1 vertices from sol. w.p. $\left(\frac{1}{\alpha}\right)^{k_1}$
3. **Sparse Case** (Low avg. degree): Number of edges incident is $\mathcal{O}(k)$. Graph has treewidth $(1 - \Omega(1))k = \gamma k$. If there exist algo. with running time β^{tw}, s.t. $\beta^\gamma \leq \alpha$, we get our $\mathcal{O}^\star(\alpha^k)$ algo.

Dense Case

For RIAFD:

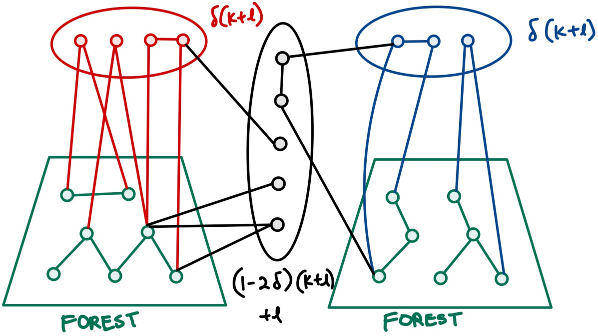
1. Apply some low degree reduction rules on the instance
2. Probabilistic reduction rule: Select vertex w.p. $\frac{\omega(v)}{\omega(V)}$
3. $\omega(v) = deg(v) - 2$, if $v \notin R$
4. $\omega(v) = 0$, if $v \in R$
5. If $deg(F) \geq \frac{4-2\epsilon}{1-\epsilon}(k+\ell)$, picked w.p. $\frac{1}{3-\epsilon}$

Sparse Case

Suppose you have an afd-set F of size k, $\overline{d} = \frac{deg(F)}{(k+\ell)} = \mathcal{O}(1)$

1. Iterative compression + $\mathcal{O}^\star(3^{\text{tw}})$ Cut&Count
2. Can obtain tree decomposition of width $(1 - 2^{-\overline{d}} + o(1))k + (2 - 2^{-\overline{d}} + o(1))\ell$

Sparse Case



$$tw(\quad) = 1 + \delta(k+l)$$

$$\therefore tw(G) = 1 + \delta(k+l) + (1-2\delta)(k+l) + l$$

$$= (1-\delta)k + (2-\delta)l + 1 \longrightarrow (1-\Omega(1))k + (2-\Omega(1))l$$

Improving dependence on ℓ

Theorem

Given a graph $G(V,E)$ with $tw(G) > 2$ and an riafd-set *F of size k, there exists a tree decomposition of width $k + \frac{3}{5.769}\ell + \mathcal{O}(\log(\ell))$ for G and it can be constructed in polynomial time.*

Proof.

- Remove F from $G \rightarrow G'(V', E')$ which is ℓ-forest
- Apply reduction rules, get $G''(V'', E'')$:
  - R_0: If there is a $v \in V'$ with $deg(v) = 0$, then remove v.
  - R_1: If there is a $v \in V'$ with $deg(v) = 1$, then remove v.
  - R_2: If there is a $v \in V'$ with $deg(v) = 2$, then contract v
- $|E''| \leq (|V''| - 1) + \ell$, $|E''| \geq 3|V''|/2$
- $|V''| \leq 2\ell$, $|E''| \leq 3\ell$

□

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Proof.

- $|V''| \leq 2\ell$, $|E''| \leq 3\ell$
- Given a graph $\mathcal{G}(V,E)$, we can obtain a tree decomposition of $\mathcal{G}$ of width at most $|E|/5.769 + \mathcal{O}(log(|V|))$ in polynomial time. Result by Kneis et al. [5]
- $tw(G') = tw(G'') \leq \frac{3}{5.769}\ell + \mathcal{O}(\log(\ell))$
- Adding F to all bags, $tw(G) \leq k + \frac{3}{5.769}\ell + \mathcal{O}(\log(\ell))$.

□

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- Combine with Iterated Compression + $\mathcal{O}^\star(3^{\text{tw}})$ Cut&Count
- Running time of $\mathcal{O}^\star(3^k 1.78^\ell)$

Conclusion

- All our results are randomized algorithms. Open: Can we design matching deterministic algorithms for these problems (including FVS)?
- There is a deterministic algorithm for Pseudoforest Deletion running in $\mathcal{O}^\star(3^k)$. Can we obtain deterministic algorithm with running time $\mathcal{O}^\star(c^k)$ where $c < 3$?
- Could we get $\mathcal{O}^\star(c^k 2^{o(\ell)})$ algorithm for Almost Forest Deletion for c possibly less than 3?

References I

Becker, A., Bar-Yehuda, R., and Geiger, D.
Randomized algorithms for the loop cutset problem.
J. Artif. Intell. Res. 12 (2000), 219–234.

Bodlaender, H. L., Ono, H., and Otachi, Y.
A faster parameterized algorithm for pseudoforest deletion.
Discret. Appl. Math. 236 (2018), 42–56.

Cygan, M., Nederlof, J., Pilipczuk, M., Pilipczuk, M., van Rooij, J. M. M., and Wojtaszczyk, J. O.
Solving connectivity problems parameterized by treewidth in single exponential time.
In *IEEE 52nd Annual Symposium on Foundations of Computer Science, FOCS 2011, Palm Springs, CA, USA, October 22-25, 2011* (2011), IEEE Computer Society, pp. 150–159.

References II

Iwata, Y., and Kobayashi, Y.
Improved analysis of highest-degree branching for feedback vertex set.
In *14th International Symposium on Parameterized and Exact Computation, IPEC 2019, September 11-13, 2019, Munich, Germany* (2019), pp. 22:1–22:11.

Kneis, J., Mölle, D., Richter, S., and Rossmanith, P.
A bound on the pathwidth of sparse graphs with applications to exact algorithms.
SIAM J. Discrete Math. 23 (01 2009), 407–427.

References III

Li, J., and Nederlof, J.
Detecting feedback vertex sets of size k in $O^*(2.7^k)$ time.
In *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020* (2020), pp. 971–989.

Li, S., and Pilipczuk, M.
An improved FPT algorithm for independent feedback vertex set.
In *Graph-Theoretic Concepts in Computer Science - 44th International Workshop, WG 2018, Cottbus, Germany, June 27-29, 2018, Proceedings* (2018), vol. 11159, Springer, pp. 344–355.

References IV

Lin, M., Feng, Q., Wang, J., Chen, J., Fu, B., and Li, W.
An improved FPT algorithm for almost forest deletion problem.
Inf. Process. Lett. 136 (2018), 30–36.